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DONALDSON'S POLYNOMIALS FOR $K3$ SURFACES

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Let M be a smooth compact simply connected four-manifold with $b_2^+ = 2p+1$, $p \geq 1$. Donaldson [5], [7] has defined polynomials $\gamma_c \in \mathrm{Sym}^d H^2(S, \mathbf{Z})$ for all $c > \frac{3}{2}(p+1)$, where $d = 4c-3(p+1)$. The polynomials are invariant under diffeomorphisms and actually provide new C^∞ invariants [5], [7]. To define these invariants choose a generic metric, g, on M and consider X_c, the Uhlenbeck compactification of the moduli space $\mathscr{M}_c$ of g-anti-self-dual connections on the SU(2) bundle on M with $c_2 = c$ [7]. There is a map $\bar{\mu}\colon H_2(M) \to H^2(X_c)$ which extends the map $\mu\colon H_2(M) \to H^2(\mathscr{M}_c)$ obtained by slant product with $-\frac{1}{4}p_2(P)$, where P is the universal SO(3) bundle over $M \times \mathscr{M}_c$. One defines

$$\gamma_c(\Gamma) = \int_{[X_c]} \underbrace{\mu(\Gamma) \cup \mu(\Gamma) \cup \cdots \cup \mu(\Gamma)}_{d \text{ times}} .$$

If M is the smooth manifold underlying a projective complex surface S, and g is the Kähler metric associated to an ample divisor H, then, by a theorem of Donaldson [4], $\mathscr{M}_c \cong M_S(H, 0, c)$, where $M_S(H, 0, c)$ is the moduli space of rank-two vector bundles E on S with $c_1(E) = 0$ and $c_2(E) = c$, μ-stable with respect to H. By passing to the algebraic-geometric situation Donaldson has proved that, for a projective surface, $\gamma_c \neq 0$, at least for big c [5]. Not much is known about Donaldson's polynomials: R. Friedman and J. Morgan have partially computed γ_c for simply connected elliptic surfaces. In particular, let S be a $K3$ surface with $c \geq 4$, $d = 4c-6 = 2n$, and q the quadratic form of S. They show that

$$\gamma_c = \frac{(2n)!}{2^n n!} q^n .$$

The aim of this paper is to give a different proof of this formula in the case where c is odd. We do this by defining a polynomial $\delta_c \in \mathrm{Sym}^d H^2(S, \mathbf{Z})$ analogous to γ_c, the difference being that instead of X_c we use the compactification of $M_S(H, 0, c)$ provided by the moduli space of semistable

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sheaves. We prove that although γ_c and δ_c are not a priori equal, in fact they are the same polynomial (we prove this only for certain polarized $K3$ surfaces and a corresponding value of c, but our arguments can be generalized to any $K3$ surface); this should be generalizable to many other surfaces. Then we compute $\delta_c(\Gamma+\overline{\Gamma})$, where Γ is the Poincaré dual of a nonzero holomorphic two-form on S; it is plausible that the method we employ can be applied to any surface. The result follows because γ_c is a multiple of a power of the quadratic form for a $K3$ surface.

Notation. Let E be a coherent torsion-free sheaf on a projective surface S, and let H be the hyperplane class on S. Then we say E is μ-stable (respectively semistable) if $\mu(F) < \mu(E)$ (respectively $\leq$) for every subsheaf $F \hookrightarrow E$, where $\mu(G) = (c_1(G)\cdot H)/\operatorname{rank}(G)$. We say E is stable (respectively semistable) if $p_F(n) < p_E(n)$ (respectively $\leq$) for all subsheaves $F \hookrightarrow E$ and all $n \gg 0$, letting $p_G(n) = \chi(G(n))/\operatorname{rank}(G)$, i.e., if E is stable (semistable) according to Gieseker and Maruyama. Both notions of stability depend on the polarization chosen, so to be precise one should always specify H. We denote by $M_S(H, c_1, c_2)$ the moduli space of rank-two locally free sheaves, E, on S, μ-stable with respect to H, with $c_1(E) = c_1$ and $c_2(E) = c_2$. We let $\overline{M}_S(H, c_1, c_2)$ be the moduli space of rank-two torsion-free sheaves, E, on S, Gieseker-Maruyama semistable with respect to H, with $c_1(E) = c_1$ and $c_2(E) = c_2$; it is a projective scheme [8], [10]. There is a natural embedding $\imath\colon M_S(H, c_1, c_2) \hookrightarrow \overline{M}_S(H, c_1, c_2)$, and $\imath(M_S(H, c_1, c_2))$ is clearly open in its closure, but a priori it need not be that $\overline{M}_S(H, c_1, c_2)$ is the closure of $\imath(M_S(H, c_1, c_2))$: there could possibly exist components all of whose points parametrize sheaves which are not locally free. When $c_1 = 0$ and $c_2 = c$, and there is no confusion about S and H, we will abbreviate $M_S(H, c_1, c_2)$ and $\overline{M}_S(H, c_1, c_2)$ to M_c and $\overline{M}_c$ respectively. Let E^{**} be the double dual of E. By the canonical sequence of E we will mean the exact sequence

$$0 \to E \to E^{**} \to Q \to 0,$$

where Q is a sheaf which naturally lives on Y, the zero-dimensional subscheme of S defined by the ideal sheaf $\operatorname{Ann} Q$. For such Q and Y we set $l(Q) = h^0(Q)$ and $l(Y) = h^0(\mathcal{O}_Y)$. In general we will denote by $[X]$ the equivalence class of an object X for an appropriate equivalence relation. So, for example, if E is an H-semistable sheaf, then $[E]$ will be a point in an appropriate moduli space, if $Z \subset S$ is a zero-dimensional subscheme, then $[Z]$ will be the corresponding point in the appropriate Hilbert scheme, etc.

1. Lemma 1. *Let S be a $K3$ surface, H a polarization on S, and E an H-semistable rank two torsion-free sheaf on S, and let $c_1(E) = 0$, $c_2(E) = c$ with c odd. Then E is stable.*

Proof. In Gieseker's notation

$$p_E(n) = \tfrac{1}{2}H^2n^2 - c/2 + 2.$$

Let $F \to E$ be a rank-one subsheaf of E. Then

$$p_F(n) = \tfrac{1}{2}H^2n^2 + (\det F \cdot H)n + \tfrac{1}{2}(\det F)^2 - c_2(F) + 2.$$

If E were semistable, there would exist F such that $p_F(n) = p_E(n)$. This is impossible because the constant coefficient of $p_F(n)$ is an integer (the intersection form is even), while the constant coefficient of $p_E(n)$ is not integer.

Corollary. *Let c be odd. If $\overline{M}_c$ is not empty, then it is smooth of dimension $4c - 6$, and there exists a universal sheaf over $S \times \overline{M}_c$.*

Proof. By the lemma, if $[E] \in \overline{M}_c$, then E is stable, hence simple. By a result of Mukai [13, Theorem 0.3], $\overline{M}_c$ is smooth at $[E]$ of dimension $4c - 6$. Again by a theorem of Mukai [13, Theorem A.6] a universal sheaf exists.

Proposition 1. *Let S be a $K3$ surface whose Picard group is generated by the ample divisor H, and let $H^2 = 2m$, and $c = 2m + 3$. Then $\overline{M}_c$ is irreducible and birational to the Hilbert scheme of zero-dimensional subschemes of S of length $4m + 3$.*

Proof. If $[E] \in \overline{M}_c$ let $F = E \otimes H$. Then $c_1(F) = 2H$ and $c_2(F) = 4m + 3$.

Claim 1. *The sheaf F fits into the exact sequence*

$$(*) \qquad 0 \to \mathcal{O}_S \to F \to I_Z(2H) \to 0,$$

where $Z \subset S$ is a zero-dimensional subscheme of length $4m + 3$.

Proof. By Riemann-Roch, $h^0(F) + h^2(F) \geq 1$; let us prove that $h^0(F) \geq 1$. By considering the canonical sequence of F we see that $h^2(F) = h^2(F^{**})$. By Serre duality, $h^2(F^{**}) = h^0(F^*)$; if $h^0(F^*) > 0$ there is an injection $\mathcal{O}_S(kH) \to F^*$, $k \geq 0$, hence an injection $\mathcal{O}_S((2+k)H) \to F^{**}$ and consequently a map $I_Z((2+k)H) \to F$ for some zero-dimensional $Z \subset S$. This clearly contradicts the stability of F, hence $h^2(F) = 0$ and $h^0(F) \geq 1$. From the stability of F it follows that any nonzero section has isolated zeros, hence it defines an injection $\mathcal{O}_S \to F$ with quotient a torsion-free rank-one sheaf $\mathcal{L}$ which is isomorphic to $I_Z(2H)$ for some zero-dimensional subscheme $Z \subset S$. Since $c_2(F) = 4m + 3$, the length of Z is $4m + 3$.

If F fits into the exact sequence $(*)$, then the following equalities hold:

(i) $h^0(F) = h^0(I_Z(2H)) + 1$.

(ii) $h^0(I_Z(2H)) + 1 = h^1(I_Z(2H))$.

(iii) $h^1(I_Z(2H)) = \dim \operatorname{Ext}^1(I_Z(2H), \mathscr{O}_S)$.

The first two equalities follow from the long exact cohomology sequences associated to $(*)$ and the exact sequence $0 \to I_Z(2H) \to \mathscr{O}_S(2H) \to \mathscr{O}_Z(2H) \to 0$, respectively. Equality (iii) follows from Serre duality.

Claim 2. *Let $Z \subset S$ be a zero-dimensional subscheme of length $4m+3$ such that, if $Z' \subset Z$ is a subscheme of length $4m+2$ with $h^0(I_{Z'}(2H)) = 0$, then there is a unique stable locally free sheaf F fitting into the exact sequence $(*)$.*

Proof. By our hypothesis $h^0(I_Z(2H)) = 0$, hence by (ii) and (iii) there is a unique nontrivial extension, F, of $I_Z(2H)$ by $\mathscr{O}_S$. Since Z satisfies the Cayley-Bacharach property relative to $|2H|$, the sheaf F is locally free. Let $0 \to \mathscr{O}_S(kH) \to F$ be a sublinebundle. Since, by (i), $h^0(F) = 1$, we must have $k \leq 0$, i.e., F is stable.

Definition 1. Let $\mathscr{H}_c$ be the Hilbert scheme of zero-dimensional subschemes of S of length $4m + 3$, and let $U_c \subset \mathscr{H}_c$ be the open subset defined by

$$U_c = \{Z | h^0(I_Z(2H)) = 0 \text{ and the corresponding extension } (*) \text{ is stable.}\}.$$

By Riemann-Roch, $h^0(2H) = 4m + 2$, hence if $Z \subset S$ is a generic zero-dimensional subscheme of length $4m + 3$, then $h^0(I_Z(2H)) = 0$ and, for any subscheme $Z' \subset Z$ of length $4m + 2$, $h^0(I_{Z'}(2H)) = 0$. By Claim 2 we conclude that U_c is not empty. Let V_c be the open subset of $\overline{M}_c$ defined by

$$V_c = \{[E] | h^0(E \otimes H) = 1\}.$$

The previous discussion defines an isomorphism $f\colon U_c \xrightarrow{\sim} V_c$ which extends to a rational map $\bar{f}\colon \mathscr{H}_c \to \overline{M}_c$.

Since V_c is open (or by a dimension count), $\bar{f}$ is a birational map between $\mathscr{H}_c$ and one component of $\overline{M}_c$. We will be done if we can prove that there are no other components of $\overline{M}_c$. By the Corollary to Lemma 1 any component has dimension $4c - 6$, hence the following claim finishes the proof of the proposition.

Claim 3. *The codimension of $\overline{M}_c \backslash V_c$ in $\overline{M}_c$ is at least two (in fact equal to two).*

Proof. Let $[E] \in \overline{M}_c$. Then $F = E \otimes H$ fits into the exact sequence $(*)$, so we have to bound the number of moduli of stable nontrivial extensions which arise from $[Z] \in \mathscr{H}_c \backslash U_c$. Let $\varphi: S \to \mathbf{P}^{4m+1}$ be the map associated to the complete linear system $|2H|$. Let $[Z] \in \mathscr{H}_c$ vary in a family $\mathscr{F}$ for which $\dim \operatorname{Ext}^1(I_Z(2H), \mathscr{O}_S)$ is constant. Then the number of moduli of F's obtained as extensions $(*)$ is at most

$$\dim \mathscr{F} + \dim \operatorname{Ext}^1(I_Z(2H), \mathscr{O}_S) - 1 - (h^0(F) - 1) = \dim \mathscr{F},$$

where we have used the equalities (i), (ii), (iii) (this is the essential point). We stratify $\mathscr{H}_c \backslash U_c$ according to the dimension of $\operatorname{span} \varphi(Z)$ and its intersection with $\varphi(S)$; since $[Z] \notin U_c$, $\dim \operatorname{span} \varphi(Z) \leq 4m$. First, assume $\operatorname{span} \varphi(Z) \cap \varphi(S)$ is zero-dimensional. Then $d = \dim(\operatorname{span} \varphi(Z)) \leq 4m - 1$. Since locally on $\mathscr{F}$ there is a subscheme $Z' \subset Z$ such that $\varphi(Z')$ spans $\varphi(Z)$ and $l(Z') = d + 1$, there is an injection $\imath: \mathscr{F} \hookrightarrow \operatorname{Hilb}^{d+1}(S)$, and hence

$$\text{number of moduli of } F\text{'s} \leq 2(d+1) \leq 8m.$$

If $\operatorname{span} \varphi(Z) \cap \varphi(S)$ is a divisor D, then either $D \in |H|$ or $D \in |2H|$. In the first case the number of moduli is $\dim |H| + 4m + 3 = 5m + 5$, and in the second it is $\dim |2H| + 4m + 3 = 8m + 4$. Since $\dim \overline{M}_c = 8m + 6$ we conclude that $\operatorname{codim}(\overline{M}_c \backslash V_c, \overline{M}_c) \geq 2$.

2. Definition 2. Let c be odd, S be a $K3$ surface, H be a polarization on S, and $\mathscr{E}$ be a universal sheaf on $S \times \overline{M}_c$. Then we set

$$\nu: H_2(S, \mathbf{Z}) \to H^2(\overline{M}_c, \mathbf{Z})$$

to be the map given by $\nu(\Gamma) = c_2(\mathscr{E})/\Gamma$.

Notice that a universal sheaf is not unique, but ν does not depend on the choice of $\mathscr{E}$. Let X_c be Uhlenbeck's compactification [7] of the moduli space of connections on the SU(2)-bundle with $c_2 = c$, anti-self-dual with respect to the Kähler metric associated to H. Then one has the extended μ-map $\bar{\mu}: H_2(S) \to H^2(X_c)$. By a theorem of Donaldson [4] X_c and $\overline{M}_c$ are two (different) compactifications of M_c. If we restrict to M_c, then $\bar{\mu}$ and ν agree. Let $C \subset S$ be a curve and restrict the universal sheaf $\mathscr{E}$ to $C \times \overline{M}_c$. Choose $L \in \operatorname{Pic}^{g-1}(C)$, where g is the genus of C, and let $p: C \times \overline{M}_c \to C$ and $q: C \times \overline{M}_c \to \overline{M}_c$ be the projections. Then applying Grothendieck-Riemann-Roch to $\mathscr{F} = \mathscr{E} \otimes p^*(L)$ and q one gets

$$\nu(C) = -c_1(q_! \mathscr{F}).$$

This has an analogue in X_c—one chooses a spin structure on C, and $q_! \mathscr{F}$ is replaced by the determinant of the twisted Dirac operator.

One can choose a representative of $\nu(C)$ as follows: let

$$\Delta(C, L)_{\text{red}} = \{[E] | h^0(\mathscr{O}_C(E \otimes L)) > 0\}.$$

Then the Poincaré dual of $\nu(C)$ is represented by a cycle $\Delta(C, L)$ supported on $\Delta(C, L)_{\mathrm{red}}$ (with positive coefficients). On the other hand, as is shown by Friedman and Morgan [7], $\Delta(C, L)$ restricted to M_c also represents $\mu(C)$. For this to make sense one has to choose L so that $\Delta(C, L)$ is a divisor (maybe empty), i.e., every component of $\overline{M}_c$ must contain a point $[E]$ such that $h^0(\mathscr{O}_C(E \otimes L)) = 0$. By a theorem of Raynaud [14] this is equivalent to $\mathscr{O}_C(E)$ being semistable. If C is an ample divisor and E is μ-stable with respect to C, then Mehta and Ramanathan [11] have shown that there exist $n > 0$ and $C' \in |nC|$ such that $\mathscr{O}_C(E)$ is stable. We will need the following stronger version due to Bogomolov [2, 11.8, Corollary 1].

Theorem (Bogomolov). *Let S be a projective surface, H an ample line bundle on S, and E an H μ-stable rank-two vector bundle over S with Chern classes c_1, c_2. Then there exists a number $k(c_1, c_2)$, depending on c_1 and c_2 but not on E, such that if $k \geq k_0$ and C is any smooth curve in $|kH|$, then $E_{|C}$ is stable.*

Definition 3. Let S, H, c be as in Definition 2, and let $d = 4c - 6 = \dim \overline{M}_c$. We define $\delta_c \in \mathrm{Sym}^d(H^2(S, \mathbf{Z})) \cong \mathrm{Sym}^d(H_2(S, \mathbf{Z})^*)$ by setting

$$\delta_c(\Gamma) = \nu(\Gamma)^d \quad \text{for } \Gamma \in H_2(S, \mathbf{Z}).$$

The polynomial δ_c depends a priori on the polarization chosen to define $\overline{M}_c$ and on the polarized $K3$ S, so whenever we want to stress this dependence we dénote it by $\delta_c(S, H)$. It is clearly analogous to Donaldson's polynomial γ_c, but it is not a priori obvious that they are equal.

Lemma 2. *Let (S, H) be a polarized $K3$ surface, let c be odd, and assume $\overline{M}_c$ is not empty. Then $\gamma_c(H) = \delta_c(S, H)(H)$.*

Proof. The proof follows Donaldson's method for proving that $\gamma_c(H) \neq 0$ [5]. Let $d = \dim \overline{M}_c = 4c - 6$. We will show that for k large enough one can choose smooth curves $C_i \in |kH|$, $i = 1, \cdots, d$, and line bundles $L_i \in \mathrm{Pic}^{g-1}(C_i)$, where g is the genus of C_i, such that the representatives $\Delta(C_i, L_i)$ of $\nu(kH)$ intersect only in M_c and the intersection is a finite set of points (a priori it could be empty, but in fact our main theorem shows it is not). Let g_H be the Kähler metric associated to the polarization H. Then, as we will see, g_H and the $\Delta(C_i, L_i)$'s define an admissible system in the terminology of Donaldson [5], hence the intersection of their restrictions to M_c computes $\gamma_c(H)$, but then, since there is no point of intersection on $\overline{M}_c \backslash M_c$, $\gamma_c(H) = \delta_c(H)$.

We introduce the following notation: $\Delta_l(C, L) = \Delta(C, L)_{|M_l}$. We also need to observe that the set $\mathscr{S} = \{F \in \mathrm{Pic}(S) | -c \leq F^2 \leq 0,\ F \cdot H = 0\}$

is finite: this follows from the Hodge index theorem and the fact that S is regular. By Bogomolov's Theorem there exists k such that if $C \in |kH|$ and $[E] \in M_l$ for $l \leq c$, then $E_{|C}$ is stable; clearly we can also assume that $|kH|$ is very ample.

Claim. *We can choose smooth curves* $C_i \in |kH|$ *and line bundles* $L_i \in \mathrm{Pic}^{g-1}(C_i)$ *for* $i = 1, \cdots, d$ *such that*

(1) *no three of the* C_i*'s intersect,*

(2) *for all* $i \leq d$, *if* $F \in \mathscr{S}$ *then* $h^0(L_i \otimes F_{|C_i}) = 0$,

(3) $\Delta_l(C_1, L_1)_{\mathrm{red}} \cap \cdots \cap \Delta_l(C_n, L_n)_{\mathrm{red}}$ *is empty or has codimension* n *for any* $n \leq d$.

Proof of claim. By induction on n. If $n = 1$ let $\{[E_1], \cdots, [E_r]\}$ be a finite set of μ-stable rank-two vector bundles on S with $c_1 = 0$ and $c_2 \leq c$ such that any irreducible component of M_l for $l \leq c$ contains at least one $[E_s]$. Let $C_1 \in |kH|$ be any smooth curve. Since $E_{s|C_1}$ is stable for all s, there exists $L_1 \in \mathrm{Pic}^{g-1}(C_1)$ such that $h^0(E_{s|C_1} \otimes L_1) = 0$ for all s; since $\mathscr{S}$ is finite we can further insure that $h^0(L_1 \otimes F_{|C_1}) = 0$. With this choice of (C_1, L_1), $\Delta_l(C_1, L_1)_{\mathrm{red}}$ is a divisor for all $l \leq c$. Now assume $(C_1, L_1), \cdots, (C_m, L_m)$ satisfy (1), (2), (3) with d replaced by m. Then let $\{[E_1], \cdots, [E_r]\}$ be a finite set as above such that for all $l \leq c$ each irreducible component of $\Delta_l(C_1, L_1)_{\mathrm{red}} \cap \cdots \cap \Delta_l(C_m, L_m)_{\mathrm{red}}$ contains at least one $[E_s]$. Furthermore, let $C_{m+1} \in |kH|$ be any smooth curve such that $C_1, \cdots, C_{m+1}$ satisfy (1). Then we choose $L_{m+1} \in \mathrm{Pic}^g(C_{m+1})$ such that $h^0(E_{s|C_{m+1}} \otimes L_{m+1}) = 0$ for all s and $h^0(L_{m+1} \otimes F_{|C_{m+1}}) = 0$ for all $F \in \mathscr{S}$. Clearly with these choices $(C_1, L_1), \cdots, (C_{m+1}, L_{m+1})$ satisfy (1), (2), (3), hence the proof is complete.

Now let us show that $\Delta(C_1, L_1)_{\mathrm{red}} \cap \cdots \cap \Delta(C_d, L_d)_{\mathrm{red}} \subset M_c$. Assume there exists

$$(*) \qquad [E] \in \Delta(C_1, L_1)_{\mathrm{red}} \cap \cdots \cap \Delta(C_d, L_d)_{\mathrm{red}}$$

with $[E] \in \overline{M}_c \backslash M_c$. Consider the canonical sequence of E,

$$0 \to E \to E^{**} \to \mathscr{Q} \to 0.$$

Let $Z \subset S$ be the zero-dimensional subscheme whose ideal sheaf is $\mathrm{Ann}\,\mathscr{Q}$, let Z_{red} be the reduced Z, and let $c_2(E^{**}) = l$. Then $c_2(E^{**}) + l(\mathscr{Q}) = c$. If $[E] \in \Delta(C_i, L_i)$, then $h^0(E^{**}_{|C_i} \otimes L_i) > 0$ or $Z_{\mathrm{red}} \cap C_i \neq \varnothing$. Since E is Gieseker-Maruyama stable, the double dual E^{**} is μ-semistable. We distinguish two cases.

First case: E^{**} *is μ-stable.* Since $[E] \notin M_c$, we have $E^{**} \ncong E$, hence $l < c$. Let $a = \#\{i | [E^{**}] \in \Delta_i(C_i.L_i)\}$ and $b = \#\{i | Z_{\mathrm{red}} \cap C_i \neq \emptyset\}$; then by $(*)$ $a + b \geq d$. From our choice of the (C_i, L_i)'s it follows that $a \leq \dim M_l = 4l - 6$. On the other hand clearly $b \leq 2(\# Z_{\mathrm{red}}) \leq 2l(\mathscr{Q}) = 2(c - l)$, hence $d \leq a + b \leq 2c + 2l - 6 < 4c - 6 = d$, which is absurd.

Second case: E^{**} *is μ-semistable but not stable.* Let F be the semistabilizing line bundle of E^{**}, i.e., $F \cdot H = 0$ and E^{**} fits into

$$(**) \qquad 0 \to F \to E^{**} \to I_W \otimes F^* \to 0,$$

where $W \subset S$ is a zero-dimensional subscheme. From $(**)$ we get that $c_2(E^{**}) = l(W) - F^2$, by the Hodge index theorem $F^2 \leq 0$, hence $-c \leq F^2 \leq 0$, i.e., $F \in \mathscr{S}$. If Z is, as above, the subscheme on which $\mathscr{Q}$ lives, then $[E] \in \Delta(C_i, L_i)$ implies that one of the following holds:

(1) $h^0(E^{**}_{|C_i} \otimes L_i) > 0$.

(2) $W_{\mathrm{red}} \cap C_i \neq \emptyset$.

(3) $Z_{\mathrm{red}} \cap C_i \neq \emptyset$.

Since $F \in \mathscr{S}$, we know that (1) cannot hold. Let α, β be the number of i's such that (2), (3) hold, respectively. Clearly $\alpha \leq 2(\# W_{\mathrm{red}}) \leq l(W) \leq 2l$ and $\beta \leq 2(c - l)$, hence $d \leq \alpha + \beta \leq 2c < 4c - 6 = d$, which is absurd.

Next we claim that the Kähler metric g_H and the $\Delta(C_i, L_i)$'s define an admissible system, as defined by Donaldson [5]. In fact we only have to notice that, by a theorem of Mukai [13, Theorem 0.3], M_l is smooth and of the expected dimension (if not empty) whatever l is; but then our choice of the (C_i, L_i)'s ensures that the $\Delta(C_i, L_i)$'s define an admissible system. By Donaldson's Proposition 3.6 [5] the intersection number $\Delta_c(C_1, L_1) \cdot \cdots \cdot \Delta_c(C_d, L_d)$ is equal to $\gamma_c(kH)$. On the other hand, since the $\Delta(C_i, L_i)$'s do not intersect in $\overline{M}_c \backslash M_c$, $\Delta_c(C_1, L_1) \cdot \cdots \cdot \Delta_c(C_d, L_d) = \delta_c(S, kH)(kH)$, hence we conclude that $\gamma_c(kH) = \delta_c(kH)$.

The following lemma is well known in the case of locally free sheaves.

Lemma 3. *Let S be a $K3$ surface, let $A \subset \mathrm{Pic}(S)$ be the subset of ample line bundles, and let $R_c = \{F \in \mathrm{Pic}(S) | -c \leq F^2 \leq 0\}$. The set of walls $W_c = \{F^{\perp} \subset \mathrm{Pic}(S) | F \in R\}$ determined by R_c partitions the ample cone $A \otimes \mathbf{R}$ into chambers. Let H_1, H_2 be polarizations on S and assume that they belong to the same open chamber of $A \otimes \mathbf{R}$. Then $\overline{M}_S(H_1, 0, c) \cong \overline{M}_S(H_2, 0, c)$.*

Proof. We must show that a sheaf E cannot be H_2-semistable and H_1 nonsemistable (then we exchange the roles of H_1 and H_2). Let

$$(*) \qquad 0 \to I_\Gamma(F) \to E \to I_{\Gamma'}(-F) \to 0$$

be an H_1 desemistabilizing sequence. Let $\gamma = h^0(\mathscr{O}_\Gamma)$ and $\gamma' = h^0(\mathscr{O}_{\Gamma'})$.

Then $c = -F^2 + \gamma + \gamma'$, hence

$$(\dagger) \qquad F^2 \geq -c.$$

Assume $F \cdot H_1 > 0$ and $F \cdot H_2 < 0$. Then by the Hodge index theorem $F^2 < 0$, and by $(\dagger)$ H_1 and H_2 cannot belong to the same chamber, impossible. If $F \cdot H_1 > 0$ and $F \cdot H_2 = 0$, again by Hodge index $F^2 < 0$, and by $(\dagger)$ and our hypothesis it is impossible. If $F \cdot H_1 = 0$ either $F = 0$ or $F^2 < 0$. By $(\dagger)$ and our hypothesis $F^2 < 0$ is impossible. If $F = 0$, since $I_\Gamma(F)$ is H_1 desemistabilizing, $-\gamma > -c/2$, but $-\gamma \leq -c/2$ since E is H_2 semistable, impossible.

Corollary. *Let S be a $K3$ surface, H a polarization on S, and c an odd number. Assume $\overline{M}_c$ is not empty, and H does not lie on a wall of W_c. Then*

$$\gamma_{c|\operatorname{Pic}(S)} = \delta_c(S, H)_{|\operatorname{Pic}(S)}.$$

Proof. Let C_H be the intersection of the open chamber containing H and $\operatorname{Pic}(S)$, and let $H_i \in C_H$. By Lemma 3 we know that $\delta_c(S, H)(H_i) = \delta_c(S, H_i)(H_i)$, and, by Lemma 2, $\delta_c(S, H_i)(H_i) = \gamma_c(H_i)$, hence $\delta_c(S, H) \times(H_i) = \gamma_c(H_i)$. The set of lines $\{[H_i] | H_i \in C_h\}$ is a Zariski dense subset of $\mathbf{P}(\operatorname{Pic}(S) \otimes \mathbf{R})$, hence the two homogeneous polynomials $\gamma_{c|\operatorname{Pic}(S)}$ and $\delta_c(S, H)_{|\operatorname{Pic}(S)}$ must be equal.

Lemma 4. *Let S be a $K3$ surface, H be a primitive polarization on S, $H^2 = 2m$, $c = 2m+3$, and $d = 4c-6$. Let $q \in \operatorname{Sym}^2(H^2(S, \mathbf{Z}))$, $h \in H^2(S, \mathbf{Z})$ be the intersection form and $c_1(H)$ respectively. Then $\delta_c(S, H)$ is a polynomial in q and h, i.e.,*

$$\delta_c(S, H) = a_0 q^{d/2} + a_1 q^{d/2-1} h^2 + \cdots + a_{d/2} h^d$$

for some rational numbers $a_0, a_1, \cdots, a_{d/2}$.

Proof. The surface S belongs to the family $\mathscr{B}$ of all $K3$ surfaces with a primitive polarization of degree H^2, which will be surfaces in a fixed $\mathbf{P}^r$, $r = h^0(S, nH) - 1$ $(n \geq 3)$. By Gieseker and Maruyama's theorem ([8], [10]), there is a relative moduli space $\mathscr{M}$ of H-semistable sheaves over $\mathscr{B}$. Let $\pi: \mathscr{M} \to \mathscr{B}$ be the projection. By Proposition 1, $\pi(\mathscr{M})$ contains the dense subset $\mathscr{B}_0 \subset \mathscr{B}$ parametrizing surfaces whose Picard group has rank one. Since π is proper, we conclude that $\pi(\mathscr{M}) = \mathscr{B}$. We would like to have a relative universal sheaf on $\mathscr{S} \times_{\mathscr{B}} \mathscr{M}$, where $\mathscr{S}$ is the universal $K3$ with a primitive polarization of degree $2m$, in order to compare the polynomials $\delta_c(S_0, H_0)$ and $\delta_c(S_1, H_1)$ for two surfaces. A relative universal sheaf might not exist, although there is one of each fiber $S \times \overline{M}_c$. But, by using a criterion of Maruyama [10, Proposition 6.10], as modified by Mukai [13, Theorem A.6], we conclude that there exists a finite covering map $\phi: \widetilde{\mathscr{B}} \to \mathscr{B}$ such that there is a "universal sheaf" on $\widetilde{\mathscr{S}} \times_{\widetilde{\mathscr{B}}} \widetilde{\mathscr{M}}$ where $\widetilde{\mathscr{S}} = \mathscr{S} \times_{\mathscr{B}} \widetilde{\mathscr{B}}$. In fact let $H_1, H_2, \cdots, H_{d-3}$ be fixed generic hyperplanes and let $\widetilde{\mathscr{B}} \subset S \times \mathscr{B}$ be defined by $\widetilde{\mathscr{B}} = \{(P, b) | P \in H_1 \cap \cdots \cap H_{d-3} \cap S\}$. By definition on $\widetilde{\mathscr{S}}$ there is a section Δ of the map to $\widetilde{\mathscr{B}}$; hence the sheaf $\mathscr{O}_\Delta$. When restricted to $S \subset \mathscr{S}$, $\mathscr{O}_\Delta$ is $\mathscr{O}_P$ and $\chi(\mathscr{O}_P(E)) = 2$; hence Mukai's criterion [13, Theorem A.6] applies in this relative case and we conclude that there exists a "universal sheaf". Let $\alpha: [0, 1] \to \widetilde{\mathscr{B}}$ be a path with end points corresponding to surfaces S_0 and S_1, and let $\alpha_*: H_2(S_0) \to H_2(S_1)$ be the natural map. Hence we conclude that $\delta_c(S_0, H_0)(v) = \delta_c(S_1, H_1)(\alpha_*(v))$. Now fix one polarized $K3$, S; then $\delta_c(S, H)$ is invariant under the action of the fundamental group of $\widetilde{\mathscr{B}}$. Since the image of $\pi_1(\widetilde{\mathscr{B}})$ in the group of isometries of $H_2(S)$ is of finite index in the subgroup fixing h, we conclude, as in [6], that $\delta_c(S, H)$ is of the given form.

Proposition 2. *Let S be a $K3$ surface, H be a primitive polarization on S of degree $2m$, and $c = 2m + 3$. Then $\delta_c(S, H) = \gamma_c$.*

Proof. By Lemma 4, $\delta_c(S, H)$ is a polynomial in q and h; on the other hand, γ_c is a polynomial in q [7], hence we can write

$$(*) \qquad \delta_c(S, H) - \gamma_c = \sum_{i=0}^{d/2} a_i q^{d/2-i} h^{2i}.$$

Let (S, H) be a polarized $K3$ surface such that $\mathrm{Pic}(S) = \mathbf{Z}[H] \oplus \mathbf{Z}[L]$, where $H^2 = 2m$, $H \cdot L = a$, $L^2 = -2$ (i.e., L is a rational curve of degree a). Such an S exists if $a > 0$. As is easily checked, whatever a is, H will not lie on any wall of W_c (the notation is as in Lemma 3), hence by the Corollary to Lemma 3 we know that

$$(**) \qquad \gamma_{c|\mathrm{Pic}(S)} = \delta_c(S, H)_{|\mathrm{Pic}(S)}.$$

Let ϕ be the polynomial on the right side of $(*)$. We claim that $(**)$ implies $\phi = 0$. Assuming $\phi \neq 0$, we will arrive at a contradiction. Write $\phi = h^{2n}\psi$, where ψ is not divisible by h, so $\psi = \sum_{i=n}^{d/2} a_i q^{d/2-i} h^{2i-2n}$ and $a_n \neq 0$. Obviously $\psi_{|\mathrm{Pic}(S)} = 0$. Let $D \in \mathrm{Pic}(S)$ be a nonzero divisor class perpendicular to H. Then $\psi(D) = a_n q(D)^{d/2-n}$ and, since $D^2 \neq 0$, we get $a_n = 0$, which is a contradiction.

Corollary. *Let S be a $K3$ surface, H be a primitive polarization on S of degree $2m$, and $c = 2m + 3$. Then $\delta_c = aq^{d/2}$.*

3. Let S be a $K3$ surface, H be a primitive polarization on S, $H^2 = 2m$, and $c = 2m + 3$. Recall from §2 that there is an isomorphism

$f\colon U_c \xrightarrow{\sim} V_c$: if $[Z] \in U_c$, then $f([Z])$ is the isomorphism class of the unique nontrivial extension of $I_Z(2H)$ by $\mathscr{O}_S$. We will therefore identify U_c and V_c. Let $Y = S \times U_c$. By a standard construction [3] there exists a universal extension

$$0 \to \mathscr{O}_Y \to \mathscr{F} \to I_{\mathscr{Z}}(p_S^*(2H) \otimes p_{U_c}^*(L)) \to 0,$$

where $\mathscr{Z} \subset S \times U_c$ is the restriction of the universal subscheme on $S \times \mathscr{H}_c$ to $S \times U_c$, p_S and P_{U_c} are the projections, and L is a line bundle on U_c. If we tensor $\mathscr{F}$ by $p_S^*(-H)$, we get a universal sheaf $\mathscr{E}$ on $S \times U_c$ and consequently on $S \times V_c$:

$$0 \to \mathscr{O}_Y(p_S^*(-H)) \to \mathscr{E} \to I_{\mathscr{Z}}(p_S^*(H) \otimes p_{U_c}^*(L)) \to 0.$$

Now choose a nonzero holomorphic two-form, ω, on S. Let $\Gamma \in H_2(S)$ be the Poincaré dual to the class $[\omega] \in H^2(S)$ represented by ω, and let $\mathrm{P.D.}(\mathscr{Z})$ be the Poincaré dual of $\mathscr{Z}$. Then

$$c_2(\mathscr{E}) = p_S^*(-c_1(H)^2) - p_S^*(H)p_{U_c}^*(L) + \mathrm{P.D.}(\mathscr{Z}).$$

Since $[\omega] \cup c_1(H) = 0$, we see that

$$c_2(\mathscr{E})/\Gamma = \mathrm{P.D.}(\mathscr{Z}))/\Gamma,$$

so that $c_2(\mathscr{E})/\Gamma$ is represented by the form obtained by integrating $p_S^*(\omega)_{|\mathscr{Z}}$ along the fibers of p_{V_c}, i.e., the push-forward of $p_S^*(\omega)_{|\mathscr{Z}}$, which we will denote by $\omega^{(n)}$, $n = 4m+3$ (since V_c is identified with U_c, we can think of V_c as a subset of $\mathscr{H}_c$, and then $\omega^{(n)}$ is the restriction of a holomorphic form on $\mathscr{H}_c$ [1]). We have proved

Lemma 5. *Let $\pi\colon \mathscr{Z} \to V_c$ be the projection and let $\omega^{(n)} \in \Gamma(\Omega^{2,0}_{U-c})$ be the push-forward of $p_S^*(\omega)_{|\mathscr{Z}}$. Then $\nu(\Gamma)$ restricted to V_c is represented by $\omega^{(n)}$.*

Lemma 6. *There exists a unique holomorphic two-form on $\overline{M}_c$, $\tau_{\overline{M}_c}(\omega)$, extending $\omega^{(n)}$ and representing $\nu(\Gamma)$.*

Proof. The point is that, by the claim following Definition 1, $\mathrm{cod}(\overline{M}_c \backslash V_c, \overline{M}_c) = 2$, hence $\omega^{(n)}$ extends holomorphically to $\tau_{\overline{M}_c}(\omega)$. Since $[\tau_{\overline{M}_c}(\omega)]_{|V_c} = \nu(\Gamma)_{|V_c}$, we conclude that they are equal on the whole $\overline{M}_c$.

Remark. We have associated to $\omega \in H^0(K_S)$ a two-form on $\overline{M}_c$. One can show that $\tau_{\overline{M}_c}(\omega)$ is (up to a multiplicative constant) the symplectic form constructed by Mukai ([12], [15]).

Theorem. *Let S be a $K3$ surface, let $c = 2m+3$ be an odd number greater than 3, and let $n = 4m+3$. Then*

$$\gamma_c = \frac{(2n)!}{2^n n!} q^n .$$

Proof. Since all $K3$ surfaces are diffeomorphic, we can assume that S has a primitive polarization, H, of degree $2m$. By Proposition 3 we know that $\gamma_c = \delta_c(S, H)$. Let $\omega \in H^0(K_S)$ be a generator; we will compute $\delta_c(\Gamma + \overline{\Gamma})$. Let $d = 8m+6 = \dim \overline{M}_c$. By Lemma 6, $\nu_c(\Gamma + \overline{\Gamma})$ is represented by $\tau_{\overline{M}_c}(\omega) + \overline{\tau_{\overline{M}_c}(\omega)}$. Then

$$\delta_c(\Gamma + \overline{\Gamma}) = \int_{\overline{M}_c} \bigwedge^d (\tau_{\overline{M}_c}(\omega) + \overline{\tau_{\overline{M}_c}(\omega)}),$$

which is equal to

$$\int_{U_c} \bigwedge^d (\omega^{(n)} + \bar{\omega}^{(n)}).$$

Now let $S_0^{(n)} \subset U_c$ be the subvariety parametrizing the Z's such that $\operatorname{supp} Z$ consists of n distinct points, let S^n be the product of n copies of S, and S_0^n be the open subvariety mapping to $S_0^{(n)}$ by the obvious map. Denote this map by f, and the ith projection by $p_i \colon S^n \to S$. Then it is clear that $f^*(\tau_{\overline{M}_c}(\omega) + \overline{\tau_{\overline{M}_c}(\omega)}) = \sum_{i=1}^n p_i^*(\omega + \bar{\omega})$, so that

$$\begin{aligned} \int_{S^n} \bigwedge^d \left(\sum_{i=1}^n p_i^*(\omega + \bar{\omega}) \right) &= (2n)! \left(\int_S \omega \wedge \bar{\omega} \right)^n \\ &= \frac{(2n)!}{2^n} \left(\int_S (\omega + \bar{\omega}) \wedge (\omega + \bar{\omega}) \right)^n . \end{aligned}$$

The first equality holds because in the wedge product the only terms which give a nonzero integral are

$$\underbrace{p_1^*(\omega) \wedge p_1^*(\bar{\omega}) \wedge p_2^*(\omega) \wedge \cdots \wedge p_n^*(\bar{\omega})}_{2n}$$

and all its permutations. Since $\deg f = n!$, we have

$$\delta_c(\Gamma + \overline{\Gamma}) = \frac{(2n)!}{2^n n!} q(\Gamma + \overline{\Gamma})^n .$$

By Proposition 2 we conclude that

$$\gamma_c = \delta_c = \frac{(2n)!}{2^n n!} q^n .$$

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